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A Hamiltonian with periodic orbits having several delays

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Abstract

In 1974 Kaplan and Yorke introduced a certain delay equation with an arbitrary number of delays, $x'(t) = -f(x(t-1)) - f(x(t-2)) - \cdots - f(x(t-n))$, conjecturing that it has periodic solutions when f is an odd homeomorphism of the reals which is differentiable at the origin and infinity. By coupling the delay equation to a vector field on $\mathbb{R}^{n+1}$ they were able to prove, when certain conditions on the derivative of f at 0 and ∞ are satisfied, one delay and two delay versions. We find that the closed orbits of the coupled vector field occur at the points of intersection of two hyperplane fields which are invariant under the flow and invariant under deformation to a linear vector field. By analyzing properties of the linear vector field we are able to give an elementary construction of $[\frac{n+1}{2}]$ periodic solutions to the general delay equation when conditions naturally extending those of Kaplan and Yorke are satisfied.

Keywords: Delay equation; Hamiltonian

1. Delay equations, the main result

Let $f : \mathbb{R} \to \mathbb{R}$ *be an odd, orientation preserving homeomorphism which is differentiable at the origin and at infinity. Let* $\alpha = f'(0)$ *and* $\beta = f'(\infty)$, *where* $0 \leqslant \alpha, \beta \leqslant \infty$.

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Suppose

$$\alpha \leqslant \frac{\pi}{n+1} \tan \frac{\pi}{2(n+1)} \leqslant \beta$$

or

$$\beta \leqslant \frac{\pi}{n+1} \tan \frac{\pi}{2(n+1)} \leqslant \alpha.$$

Then the delay equation

$$x'(t) = -f(x(t-1)) - f(x(t-2)) - \cdots - f(x(t-n))$$

has a periodic solution of period $2(n+1)$.

To prove this result we find closed orbits for a certain family of vector fields on $\mathbb{R}^{n+1}$ which are coupled to periodic solutions of delay equations with n delays. This is the approach used by Kaplan and Yorke [2], to prove one delay and two delay versions of the above result.

More specifically, to construct a periodic solution, $(x, y, z, w, \ldots)$, to the delay equation for a given f we first couple it to a Hamiltonian vector field on $\mathbb{R}^{n+1}$ all of whose level hypersurfaces are homeomorphic to S^n. We deform the vector field to one associated to a linear f in such a way that closed orbits are preserved. The linear Hamiltonian vector field on $\mathbb{R}^{n+1}$ has, on each hypersphere, $[\frac{n+1}{2}]$ closed orbits which appear at the intersection of the eigenspaces with the level spheres. Each closed orbit has a delay; this is a time δ, so that $x(t-\delta) = y(t), y(t-\delta) = z(t), \ldots$. This delay varies from one eigenspace to the other; it is $\frac{m}{2(n+1)}$ times the period, where m is odd, $1 \leqslant m \leqslant n$, but this ratio is invariant under deformation. With appropriate conditions on the slope of f at 0 and ∞ we will see that we can obtain solutions of delay 1.

We obtain the following general result.

Let $f : \mathbb{R} \to \mathbb{R}$ *be an odd, orientation preserving homeomorphism, differentiable at the origin and infinity. Let* $\alpha = f'(0)$ *and* $\beta = f'(\infty)$, *where* $0 \leqslant \alpha, \beta \leqslant \infty$. *Suppose* m *is odd*, $1 \leqslant m \leqslant n$, *and*

$$\alpha \leqslant \frac{m\pi}{n+1} \tan \frac{m\pi}{2(n+1)} \leqslant \beta$$

or

$$\beta \leqslant \frac{m\pi}{n+1} \tan \frac{m\pi}{2(n+1)} \leqslant \alpha.$$

Then the delay equation

$$x'(t) = -f(x(t-1)) - f(x(t-2)) - \cdots - f(x(t-n))$$

has a periodic solution of period $2(n+1)/m$.

2. History

We give a brief account of research related to the Kaplan–Yorke delay equation.

In 1978 Nussbaum, [7], gave a functional analysis proof of the Kaplan–Yorke Conjecture. He formulated a more general delay equation whose solutions correspond to fixed points of a differentiable, asymptotically linear map in Banach Space. Under appropriate conditions, made to accommodate the more general equation, he was able to apply a fixed point theorem of Krasnosel'skii to prove the existence of periodic solutions with period $2(n+1)$. In Section 9 we compare those conditions to ours in the case of three delays.

The coupled vector field for one and two delays has only closed orbits. Kaplan and Yorke's proof fails to extend to three delays mainly because the coupled vector field lies on the 3-sphere where closed orbits are difficult to find. Coincidentally, the year Kaplan and Yorke's paper appeared was also the year that P. Schweitzer gave the first counterexample to what was a long-standing conjecture of Seifert that all vector fields on the 3-sphere have closed orbits. The first counterexample was C^1, but now counterexamples are known even for real analytic vector fields.

In 1978, in one of the early works of its kind, [8], Rabinowitz showed that Hamiltonian vector fields, with certain prescribed level surfaces, have closed orbits. Not all Hamiltonians do; the search for conditions which ensure their existence remains an area of active research today. See [1] for a bibliography which includes the main results up to 1995 in this field.

Li and collaborators began in the mid-1990s to exploit the closed orbit theorems for Hamiltonians in order to "connect" those solutions to periodic solutions of the Kaplan–Yorke delay equation. For example, in [3–6], they consider a more general equation, one in which the delays are real numbers. Typically, in their work, the proof of existence of periodic solutions depends on algebraic constraints on the delays as well as differentiability of the function f. In Section 9 we compare their results more explicitly to ours.

We point out that in our paper, even though we refer to our coupled vector field as Hamiltonian, it is only for the sake of identifying it as such. We make use of no known results or constructions involving the existence of periodic orbits for Hamiltonians.

3. A family of Hamiltonian vector fields on $\mathbb{R}^{n+1}$

Let $f : \mathbb{R} \to \mathbb{R}$ be an odd, orientation preserving homeomorphism which is differentiable at the origin and infinity. Consider the Hamiltonian $F : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by

$$F(x, y, z, w, \ldots) = \int_0^x f(s)\,ds + \int_0^y f(s)\,ds + \int_0^z f(s)\,ds + \int_0^w f(s)\,ds + \cdots$$

Let J be the $(n+1)\times(n+1)$ symplectic matrix

$$\begin{pmatrix} 0 & -1 & -1 & \dots & -1 & -1 \\ 1 & 0 & -1 & \dots & -1 & -1 \\ 1 & 1 & 0 & \dots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & 0 & -1 \\ 1 & 1 & 1 & \dots & 1 & 0 \end{pmatrix}.$$

Define Hamiltonian vector fields on $\mathbb{R}^{n+1}$ by $v(F) = J \cdot \nabla F$, where ∇ denotes the gradient. Note that if $i : \mathbb{R} \to \mathbb{R}$ is the identity map then $I(x, y, z, w, \dots) = \frac{1}{2}(x^2 + y^2 + z^2 + w^2 + \cdots)$, and $v(I)$ is the matrix J considered as a linear vector field. More generally let l be the linear map with slope k. Let L be the corresponding Hamiltonian. Then $v(L)$ is the linear vector field kJ.

Given a vector field v, and a plane field K, we will say that K is v-invariant if at each point x the vector $v(x)$ is in K_x.

We construct, for each f, a pair of $v(F)$-invariant hyperplane fields.

One can check directly that the system determined by the Hamiltonian vector field $v(F) = J \cdot \nabla F$ satisfies the two equations

$$f(x_1)x_1' + \cdots + f(x_{n+1})x_{n+1}' = 0,$$

$$f(x_1)x_2' + f(x_2)x_1' + \cdots + f(x_n)x_{n+1}' + f(x_{n+1})x_n' - f(x_{n+1})x_1' - f(x_1)x_{n+1}' = 0.$$

In differential notation these equations are

$$f(x_1)\,dx_1 + \cdots + f(x_{n+1})\,dx_{n+1} = 0,$$

$$\begin{aligned}&(f(x_2) - f(x_{n+1}))\,dx_1 + (f(x_1) + f(x_3))\,dx_2 + \cdots \\ &\quad + (f(x_{n-1}) + f(x_{n+1}))\,dx_n + (f(x_n) - f(x_1))\,dx_{n+1} = 0.\end{aligned}$$

The solutions are plane fields on $\mathbb{R}^{n+1}$, K_1 and K_2. The first is the tangent plane field to the level sets of F. The latter plane field is not integrable in general. But if f is linear, K_2 is the tangent plane field to a family of hyperboloids. This will be discussed further in Sections 4 and 5.

All the $c \neq 0$ constant level hypersurfaces of F, M_F^c, are radially diffeomorphic to the n-sphere S^n, as we now observe.

For each $(x, y, z, w, \ldots) \in M_F^c$ there is a unique $h(x, y, z, w, \ldots) \in \mathbb{R}$ so that $h(x, y, z, w, \ldots) \cdot (x, y, z, w, \ldots) \in M_I^1$. The function h satisfies

$$\int_0^x f + \int_0^y f + \int_0^z f + \int_0^w f + \cdots$$
$$= c(x^2 + y^2 + z^2 + w^2 + \cdots) \cdot h^2(x, y, z, w, \ldots),$$

and is smooth, and non-singular away from the origin. The map

$$D(x, y, z, w, \ldots) = h(x, y, z, w, \ldots) \cdot (x, y, z, w, \ldots)$$

is a diffeomorphism from M_F^c to M_L^1.

Such Hamiltonian vector fields always have closed orbits, but we will not make use of any of the known existence results. We will find the closed orbits explicitly as deformations of those coming from linear Hamiltonian vector fields. They will appear at the points of tangency of the two hyperplane fields K_1 and K_2.

4. The linear Hamiltonian vector fields $J \cdot \nabla L$

We now take a close look at the linear case. We have two objectives. The first is to determine the relationship between the slope k of l, and the delay, (defined below), of each corresponding closed orbit of the associated linear Hamiltonian vector field. The second is to describe the closed orbits in the linear case geometrically as the points of tangency of two families of invariant hypersurfaces of the flow. All the arguments in this section involve only linear algebra.

Recall, a non-trivial linear map, $l : \mathbb{R} \to \mathbb{R}$, of slope k, gives rise to a Hamiltonian L, which defines a linear vector field $v(L)$ on $\mathbb{R}^{n+1}$.

The eigenvalues are purely imaginary if $n + 1$ is even, and there is an additional eigenvalue of 0 if $n+1$ is odd. Each eigenspace has only periodic orbits and the period is constant on an eigenspace. So, when restricted to any level sphere, the linear vector field has $[\frac{n+1}{2}]$ closed orbits.

Now we determine values of the slope k for which the vector field $v(L)$ has a periodic solution $(x(t), y(t), z(t), w(t), \ldots)$ with *delay one*; by that we mean

$$x(t-1) = y(t), \quad x(t-2) = z(t), \quad x(t-3) = w(t), \ldots, x\left(t - \left[\frac{n+1}{2}\right]\right) = -x(t).$$

Suppose $x(t) = a\cos\lambda t + b\sin\lambda t$. Substitution in

$$x' = -kx((t-1)) - kx((t-2)) - \cdots - kx((t-n))$$

yields the two equations

$$b\lambda = -k[a(\cos\lambda + \cos 2\lambda + \cdots + \cos n\lambda) - b(\sin\lambda + \sin 2\lambda + \cdots + \sin n\lambda)],$$

$$-a\lambda = -k[a(\sin\lambda + \sin 2\lambda + \cdots + \sin n\lambda) + b(\cos\lambda + \cos 2\lambda + \cdots + \cos n\lambda)].$$

We find that, provided $a^2 = b^2$,

$$\cos\lambda + \cos 2\lambda + \cdots + \cos n\lambda = 0$$

and

$$\lambda = k(\sin\lambda + \sin 2\lambda + \cdots + \sin n\lambda).$$

Note $\lambda = \pi/(n+1)$ solves the first of the above two equations; in fact so does $\lambda = m\pi/(n+1)$ for any odd m. For any odd m there are distinct solutions λ to the second equation provided $1 \leqslant m \leqslant n$.

Therefore, the values of k for which there are periodic solutions of delay 1 to the vector field $v(L)$ on $\mathbb{R}^{n+1}$ are

$$k_n^m = \frac{m\pi}{n+1} \cdot \frac{1}{\sin\frac{m\pi}{n+1} + \sin\frac{2m\pi}{n+1} + \cdots + \sin\frac{mn\pi}{n+1}}.$$

The summation in the denominator, $\sin\frac{m\pi}{n+1} + \sin\frac{2m\pi}{n+1} + \cdots + \sin\frac{mn\pi}{n+1}$, is equal to $\cot\frac{m\pi}{2(n+1)}$.

The above computation determines the eigenvalues of J. Taking $k = 1$, the imaginary parts of the eigenvalues are, as m varies through odd integers between 1 and n,

$$\lambda_n^m = \sin\frac{m\pi}{n+1} + \sin\frac{2m\pi}{n+1} + \cdots + \sin\frac{mn\pi}{n+1}.$$

There is an additional eigenvalue of 0 when n is even.

We carry out some explicit calculations for small n.

$n = 1:$ $k = \pi/2$ gives a delay 1 solution with period 4.
$n = 2:$ $k = \pi/(3\sqrt{3})$ gives a delay 1 solution with period 6.
$n = 3:$ $k = \pi/(4(\sqrt{2}+1))$ and $k = 3\pi/(4(\sqrt{2}-1))$ give delay 1 solutions with periods 8 and 8/3, respectively.
$n = 4:$ $k = \pi/(5\sqrt{5+2\sqrt{5}})$ and $k = 3\pi/(5\sqrt{5-2\sqrt{5}})$ give delay 1 solutions with periods 10 and 10/3, respectively.
$n = 5:$ $k = \pi/(6(2+\sqrt{3}))$, $k = 3\pi/6$ and $k = 5\pi/(6(2-\sqrt{3}))$ give delay 1 solutions with periods 12, 4, and 12/5, respectively.

We now turn our attention to the eigenspaces of J.

We consider the following two hypersurfaces, S_1, and S_2, of $\mathbb{R}^{n+1}$.

$$x_1^2 + \cdots + x_{n+1}^2 = c_1,$$

$$x_1x_2 + x_2x_3 + \cdots + x_nx_{n+1} - x_{n+1}x_1 = c_2.$$

These are invariant under the flow, for the system satisfies

$$x_1x_1' + \cdots + x_{n+1}x_{n+1}' = 0,$$

$$x_1x_2' + x_2x_1' + \cdots + x_nx_{n+1}' + x_{n+1}x_n' - x_{n+1}x_1' - x_1x_{n+1}' = 0.$$

In differential notation these equations are

$$x_1\,dx_1 + \cdots + x_{n+1}\,dx_{n+1} = 0,$$

$$(x_2 - x_{n+1})\,dx_1 + (x_1 + x_3)\,dx_2 + \cdots + (x_{n-1} + x_{n+1})\,dx_n + (x_n - x_1)\,dx_{n+1} = 0.$$

The solutions are plane fields on $\mathbb{R}^{n+1}$, K_1 and K_2, which are the respective tangent plane fields to S_1 and S_2.

We will show the following.

The set of points in $\mathbb{R}^{n+1}$ where K_1 agrees with K_2 is the union of the eigenspaces of J.

Geometrically, the closed orbits occur as points at which the spheres S_1 are tangent to the hyperboloids S_2.

We look for the set of points, $\{(x_1, \ldots, x_{n+1})\}$, and the values λ which satisfy

$$\begin{gathered}
\lambda x_1 = x_2 - x_{n+1}, \\
\lambda x_2 = x_1 + x_3, \\
\vdots \\
\lambda x_n = x_{n-1} + x_{n+1}, \\
\lambda x_{n+1} = x_n - x_1.
\end{gathered}$$

The values λ are eigenvalues, and the points $\{(x_1, \ldots, x_{n+1})\}$ are eigenvectors of the following $(n+1) \times (n+1)$ matrix K:

$$\begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 & -1 \\ 1 & 0 & 1 & \ldots & 0 & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 & 0 \\ 0 & 0 & 0 & \ldots & 1 & 0 & 1 \\ -1 & 0 & 0 & \ldots & 0 & 1 & 0 \end{pmatrix}.$$

This matrix is symmetric so all eigenvalues are real. If $n+1 = 2m$, or $n+1 = 2m+1$ we claim there are m distinct eigenvalues each associated to a 2-dimensional eigenspace. Each closed orbit of the linear Hamiltonian vector field J lies in one of these planes so the eigenspaces of J and K are the same.

We determine the eigenvalues and eigenspaces of K. Define, recursively, polynomials P_j in λ by

$$P_0 = 0, \quad P_1 = 1, \quad P_j = \lambda P_{j-1} - P_{j-2}.$$

We will show first that any simultaneous solution of $P_{n+1} = 0$ and $P_n = 1$ is an eigenvalue.

Consider the matrix

$$\begin{pmatrix} -\lambda & 1 & 0 & \ldots & 0 & 0 & -1 \\ 1 & -\lambda & 1 & \ldots & 0 & 0 & 0 \\ 0 & 1 & -\lambda & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -\lambda & 1 & 0 \\ 0 & 0 & 0 & \ldots & 1 & -\lambda & 1 \\ -1 & 0 & 0 & \ldots & 0 & 1 & -\lambda \end{pmatrix}.$$

We find the values of λ which make it singular. We row reduce using the leading 1's in rows 2 through n to successively eliminate entries, in row 1, and in row $n+1$, and then we move the top row to the next to last row to obtain the following matrix K':

$$\begin{pmatrix} 1 & -\lambda & 1 & \dots & 0 & 0 & 0 \\ 0 & 1 & -\lambda & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -\lambda & 1 \\ 0 & 0 & 0 & \dots & 0 & -P_{n+1} & P_n - 1 \\ 0 & 0 & 0 & \dots & 0 & -(P_n - 1) & \lambda(P_n - 1) - P_{n+1} \end{pmatrix}.$$

Any common root α of $P_{n+1} = 0$ and $P_n = 1$ annihilates exactly the last two rows. Therefore α is an eigenvalue and has a 2-dimensional eigenspace.

Now we claim that these double roots exhaust all the eigenvalues, except when $n+1$ is odd in which case there is a single remaining eigenvalue of multiplicity 1.

Computing the determinant of the above matrix yields the following formula for the characteristic equation of K:

$$P_{n+1}^2 - \lambda(P_{n+1})(P_n - 1) + (P_n - 1)^2 = 0.$$

Solving for $P_n - 1$ in terms of P_{n+1} gives

$$P_n - 1 = P_{n+1} \cdot \left[\frac{\lambda \pm \sqrt{\lambda^2 - 4}}{2}\right].$$

Let us look for eigenvalues α which are *not* of multiplicity two. In this case neither $P_{n+1}(\alpha)$ nor $(P_n - 1)(\alpha)$ can be zero, for as we have seen such an α has multiplicity 2. Furthermore, from the form of K', we see that the only other possible multiplicity is 1.

Again, assuming $P_{n+1}(\alpha) \neq 0$ and $(P_n - 1)(\alpha) \neq 0$, it follows that $\alpha \pm \sqrt{\alpha^2 - 4} = \alpha$, for there cannot be two distinct values of $(\lambda \pm \sqrt{\lambda^2 - 4})/2$ which make the above equation hold for a fixed $\lambda = \alpha$. Therefore, if there is an eigenvalue of multiplicity 1 it must be either $\lambda = 2$ or $\lambda = -2$.

Now by direct computation $P_n(2) = n$, and $P_n(-2) = (-1)^{n+1} n$, so, by the above formula, neither 2 nor -2 is an eigenvalue when $n + 1$ is even and $\lambda = -2$ is an eigenvalue when $n + 1$ is odd. All the remaining eigenvalues have multiplicity 2 and have 2-dimensional eigenspaces.

For example, if $n + 1 = 4$, $P_3 = \lambda^2 - 1$, $P_4 = \lambda^3 - 2\lambda$, so that the eigenvalues are $\{\sqrt{2}, \sqrt{2}, -\sqrt{2}, -\sqrt{2}\}$. The eigenvectors, (P_1, P_2, P_3, P_4), after normalization, are $1/2 \cdot (1, \sqrt{2}, 1, 0)$, and $1/2 \cdot (1, -\sqrt{2}, 1, 0)$. (These provide initial points for the two closed orbits of J on the unit sphere.)

5. The non-linear Hamiltonian vector fields $J \cdot \nabla F$

We consider the Hamiltonian for a general f.

We have constructed two hyperplane fields on $\mathbb{R}^{n+1}$, K_1 and K_2, which are the respective kernels of

$$f(x_1)\,dx_1 + \cdots + f(x_{n+1})\,dx_{n+1},$$

and

$$\begin{aligned}&(f(x_2) - f(x_{n+1}))\,dx_1 + (f(x_1) + f(x_3))\,dx_2 + \cdots \\ &\quad + (f(x_{n-1}) + f(x_{n+1}))\,dx_n + (f(x_n) - f(x_1))\,dx_{n+1}.\end{aligned}$$

We will determine the set of points in $\mathbb{R}^{n+1}$ where $K_1 = K_2$. So we look for points, $\{(x_1, \ldots, x_{n+1})\}$, and values λ which satisfy

$$\begin{gathered}\lambda f(x_1) = f(x_2) - f(x_{n+1}),\\ \lambda f(x_2) = f(x_1) + f(x_3),\\ \vdots\\ \lambda f(x_n) = f(x_{n-1}) + f(x_{n+1}),\\ \lambda f(x_{n+1}) = f(x_n) - f(x_1).\end{gathered}$$

We have solved this system of equations in the linear case, that is when $f = i$. If the point $(p_1, \ldots, p_{n+1})$ solves the linear system then $(f^{-1}(p_1), \ldots, f^{-1}(p_{n+1}))$ solves for general f.

Let $\triangle f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ denote the map $\triangle f(x_1, \ldots x_{n+1}) = (f(x_1), \ldots, f(x_{n+1}))$.

Let $n+1 = 2m$, or $n+1 = 2m+1$. If $P_1, \ldots, P_m$ are the eigenspaces in the linear case, then $\{(\triangle f)^{-1}(P_1), \ldots, (\triangle f)^{-1}(P_m)\}$ are invariant spaces in the general case.

We shall refer to these invariant spaces as eigensurfaces. Each is the image of a standard plane under a homeomorphism of $\mathbb{R}^{n+1}$. Each eigensurface is the union of closed orbits of the corresponding Hamiltonian vector field.

So, given f, and the associated Hamiltonian F, to find the closed orbits of $v(F) = J \cdot \nabla F$, the eigenspaces $P_1, \ldots, P_m$ are found and then the eigensurfaces $\{(\triangle f)^{-1}(P_1), \ldots, (\triangle f)^{-1}(P_m)\}$ are intersected with the level spheres M_F^c.

6. Deformation of Hamiltonians

Let $f : \mathbb{R} \to \mathbb{R}$ be an odd homeomorphism of the reals which is differentiable at the origin and infinity, $F : \mathbb{R}^{n+1} \to \mathbb{R}$ the associated Hamiltonian, and $v(F) = J \cdot \nabla F$, the associated Hamiltonian vector field.

Given $f : \mathbb{R} \to \mathbb{R}$, we construct a homotopy from f to i by

$$h_t = tf + (1-t)i \quad 0 \leqslant t \leqslant 1.$$

For each fixed t the function h_t is itself an odd homeomorphism of the reals which is differentiable at the origin and infinity.

The Hamiltonian H_t associated to h_t is $H_t = tF + (1-t)I$.

The Hamiltonian vector field $v(H_t)$ associated to H_t is $v(H_t) = J \cdot \nabla H_t$, which is the same as $tv(F) + (1-t)v(I)$.

Fix $c > 0$, and let $M^c_{H_t}$ denote the level set $H_t = c$. The parameter t defines a deformation of manifolds each of which is radially diffeomorphic to a sphere. For each t we intersect $\{(\triangle h_t)^{-1}(P_1), \ldots, (\triangle h_t)^{-1}(P_m)\}$ with $M^c_{H_t}$. The parameter t deforms each closed orbit of the linear Hamiltonian vector field, $v(L)$, to a closed orbit of $v(F)$.

7. Periods and delays

We now show that a periodic orbit of the Hamiltonian vector field determines a periodic solution to a coupled delay equation, with some delay, δ.

Fix a Hamiltonian vector field $v(F) = J \cdot \nabla(F)$.

The linear map $T_{2m}(x, y, z, w, \ldots) = (y, z, w, \ldots, -x)$ maps solution curves to solution curves. In the linear case T_{2m} maps each closed solution to itself. Moreover T_{2m} acts continuously under the deformation, so it must map each of the m distinguished closed solutions to itself in general. The coordinates of the solution then satisfy

$$(x(t-\delta), y(t-\delta), z(t-\delta), w(t-\delta), \ldots) = (y(t), z(t), w(t), \ldots, -x(t)),$$

for some δ. Therefore $y(t) = x(t-\delta), z(t) = x(t-2\delta), w(t) = x(t-3\delta), \ldots$ and, as substitution in a given Hamiltonian vector field $v(F)$ shows, $x(t)$ solves the delay equation

$$x'(t) = -f(x(t-\delta)) - f(x(t-2\delta)) - f(x(t-3\delta)) - \cdots - f(x(t-n\delta)).$$

We have proved the following

Let $f : \mathbb{R} \to \mathbb{R}$ be an odd, orientation preserving, homeomorphism which is differentiable at the origin and infinity. Then for some δ the delay equation

$$x'(t) = -f(x(t-\delta)) - f(x(t-2\delta)) - f(x(t-3\delta)) - \cdots - f(x(t-n\delta))$$

has a periodic solution.

The period and the delay vary continuously under deformation. Moreover the delay is a fraction of the period which must remain constant under deformation. In the linear case the delay is $m/(2(n+1))$ times the period, where $m = 1, 3, \ldots, n$. So, in general, the delay is $m/(2(n+1))$ times the period where m depends on the particular closed curve.

8. Eigensurfaces and delay 1 solutions

Consider a level surface M_F^c. For each c we deform M_F^c to the level surface of a linear Hamiltonian vector field, say M_I^c. As noted above we have m closed orbits on M_I^c, one of which has delay $1/(2(n+1))$th the period, another $3/(2(n+1))$th the period, and so on. As c varies these orbits fill the m distinct eigenspaces of $v(I)$. Under the deformation the union of the closed orbits determine homeomorphic images of the m eigenspaces, one of which again has the property that the delay is $1/(2(n+1))$th the period, another that the delay is $3/(2(n+1))$th the period, etc. So for a general $v(F)$ we have a well-defined $k/(2(n+1))$-eigensurface, where $k = 1, 3, \ldots, n$. To insure the existence of a solution with period $2(n+1)$ and delay 1, the slope of f at 0 and ∞, as well as the eigensurface, must be chosen appropriately.

We now fix f and vary c. On each eigensurface, as the parameter c approaches 0, the periods of the solutions to the Hamiltonian vector field $v(F) = J \cdot \nabla(F)$ continuously approach the periods of the linear system $v(L_0)$, where l_0 has slope α at 0, and likewise as c approaches ∞ the periods vary continuously and approach the periods of the linear system $v(L_\infty)$ where l_∞ has slope β at ∞.

Therefore, recalling our analysis of the linear case, either inequality

$$\alpha \leqslant k_n \leqslant \beta \quad \text{or} \quad \beta \leqslant k_n \leqslant \alpha$$

will insure that for some c the vector field $v(F)$ has a solution with period $2(n+1)$ on the $1/(2(n+1))$-eigensurface. And, in general, either inequality

$$\alpha \leqslant k_n^m \leqslant \beta \quad \text{or} \quad \beta \leqslant k_n^m \leqslant \alpha$$

will insure that for some c the vector field $v(F)$ has a solution with period $2(n+1)/m$ on the $m/(2(n+1))$-eigensurface. This produces a solution with delay 1 for each odd $m = 1, 3, \ldots,$ where $m \leqslant n$, as required.

9. Three delays

First let us look at Nussbaum's result. To make a comparison more explicit let us analyze a special case, one considered by Nussbaum in [7]. Assume that $f : \mathbb{R} \to \mathbb{R}$ is an odd, orientation preserving homeomorphism which is differentiable at the origin and at infinity, with $f'(0) = 1$, and $f'(\infty) = 0$.

Consider the delay equation

$$x'(t) = -\alpha f(x(t-1)) - \alpha f(x(t-2)) - \cdots - \alpha f(x(t-n)),$$

for $\alpha > 0$.

Nussbaum deduces, Corollary 2, p. 146, from his main theorem that when $n = 3$, and except for certain values of α, there is a period 8 solution for $\alpha > \frac{\pi}{4(\sqrt{2}+1)}$.

Let us comment on the papers of Li et al. As already mentioned the authors make use of results on the existence of closed orbits for certain Hamiltonian vector fields, which require certain restrictions on f. More involved algebraic hypotheses enter their work, since they are incorporating general delays: $r_1, r_2, \ldots,$ not just integral delays: $1, 2, \ldots$. This leads to constraints on the relationship between the r_i's and the corresponding periods of the solutions. In applying their theorems it is more natural to fix the period and look for the delays of the corresponding solutions. To compare with the case considered above, let us do this for $n = 3$ assuming that $f : \mathbb{R} \to \mathbb{R}$ is an odd, orientation preserving differentiable homeomorphism which is differentiable at infinity, with $f'(0) = 1$, and $f'(\infty) = 0$. Let us look for period 8 solutions to

$$x'(t) = -\alpha f(x(t-r_1)) - \alpha f(x(t-r_2)) - \alpha f(x(t-r_3)),$$

for $\alpha > 0$.

An application of Theorem 1(i) of [3] yields two periodic solutions of period 8, but with delays $r_1 = 9, r_2 = 18, r_3 = 27$. These occur when $\alpha > \frac{\pi(1+\sqrt{2})}{4} \approx 1.9$.

Our theorem implies that for all $\alpha > \frac{\pi}{n+1} \tan \frac{\pi}{2(n+1)}$ the delay equation

$$x'(t) = -\alpha f(x(t-1)) - \alpha f(x(t-2)) - \cdots - \alpha f(x(t-n))$$

has a periodic solution of period $2(n+1)$. Furthermore, for $\alpha > \frac{m\pi}{n+1} \tan \frac{m\pi}{2(n+1)}$, and m odd, $m \leqslant n$, there is a periodic solution of period $2(n+1)/m$. Note, $\tan \frac{n\pi}{2(n+1)} = \cot \frac{\pi}{2(n+1)}$ produces the largest of the values $\frac{m\pi}{n+1} \tan \frac{m\pi}{2(n+1)}$, so if n is odd, then the condition $\alpha > \frac{n\pi}{n+1} \cot \frac{\pi}{2(n+1)}$ yields solutions with periods $2(n+1), 2(n+1)/3, \ldots, 2(n+1)/n$.

In particular, when $n = 3$ our results yield for $\alpha > \frac{\pi}{4(\sqrt{2}+1)} \approx .33$ a solution with period 8, and for $\alpha > \frac{3\pi}{4(\sqrt{2}-1)} \approx 5.7$ an additional solution with period 8/3.

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